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Note

A Second-Order Property of Spline Functions with One Free Knot

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We are concerned with an approximation problem by polynomial spline functions with one free knot. Our main concern is a second-order property of the problem with respect to the knot. We show that every spline function satisfying Braess's alternation condition is nearly optimal. © 1994 Academic Press, Inc.

1. INTRODUCTION

We deal with a Tchebycheff approximation problem of a prescribed continuous function $f(t)$ by polynomial spline functions with one free knot:

$$\text{minimize} \qquad S(x) := \max\{|f(t) - F(x,t)|;\, t \in [0,1]\}, \tag{1}$$

$$F(x,t) = \sum_{j=0}^{n} x_j t^j + x_{n+1}(t-\xi)_+^n,$$

$$x = (x_0, x_1, \ldots, x_{n+1}, \xi) \in R^{n+3}. \tag{2}$$

In the following, we put $0 = \xi_0 \leq \xi_1 = \xi \leq \xi_2 = 1$ and we assume that $n \geq 3$. Braess [1, Cor. 3.5] gave a necessary condition for $F(x^*, t)$ to be a local best approximation ($S(x^*) \leq S(x)$ on some neighborhood of x^*). His condition asserts that the error function $r(t) := f(t) - F(x^*, t)$ alternates at least $n + 2(q-p) - 1$ times in some interval $[\xi_p, \xi_q]$. The aim of this paper is to show that every spline function which satisfies Braess' condition is nearly optimal in the sense that

$$0 \leq S''(x^*; y) \leq +\infty \qquad \text{for all } y \in R^{n+3} \text{ s.t. } S'(x^*; y) = 0, \tag{3}$$

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where $S'(x^*; y)$ denotes the directional derivatives of $S(x)$ in the direction y and

$$S''(x^*; y) := \lim_{\theta \to +0} \frac{S(x^* + \theta y) - S(x^*) - \theta S'(x^*; y)}{\theta^2}. \tag{4}$$

When the limit in (4) does not exist, we denote the upper limit by $\bar{S}''(x^*; y)$.

2. Second-Order Property

First, we explain our notation. We denote by F_x and F_{xx} the gradient vector and the Hesse matrix of F w.r.t. x, respectively. We denote by $T(x)$ the set of all extreme points of the error function, that is, $T(x^*) := \{t \in [0, 1]; |r(t)| = S(x^*)\}$. A vector y is said to be critical if $S'(x^*; y) = 0$.

Next, we remark that Braess's condition is equivalent to that there exist extreme points $0 \le t_1 \le \cdots \le t_{n+4} \le 1$ and $\lambda_1, \ldots, \lambda_{n+4} \ge 0$ not all zero such that

$$\sum_{i=1}^{n+4} \lambda_i \sigma(t_i) F_x(x^*, t_i) = 0, \tag{5}$$

where $\sigma(t)$ denotes the sign of the error function. This fact can be proved by Carathéodory's theorem, Cramer's formula and Karlin–Ziegler's theorem [5, Thm. 1 and 2].

Theorem 1. *Suppose that condition* (5) *is satisfied at* x^*. *Then*

$$y^{\mathrm{T}}\left(\sum_{i=1}^{n+4} \lambda_i \sigma(t_i) F_{xx}(x^*, t_i)\right) y = 0 \tag{6}$$

for any critical direction $y \in R^{n+3}$.

Proof. It follows from (2) that condition (5) amounts to

$$\begin{aligned} &\sum_{i=1}^{n+4} \lambda_i \sigma(t_i) t_i^k = 0, \; k = 0, \ldots, n; \\ &\sum_{i=1}^{n+4} \lambda_i \sigma(t_i)(t_i - \xi)_+^k = 0, \qquad k = n-1, n. \end{aligned} \tag{7}$$

Hence we have

$$y^{\mathrm{T}}\left(\sum_{i=1}^{n+4}\lambda_i\sigma(t_i)F_{xx}(x^*,t_i)\right)y = n(n-1)x_{n+1}^*y_{n+2}^2\sum_{i=1}^{n+4}\lambda_i\sigma(t_i)(t_i-\xi)_+^{n-2} \tag{8}$$

for any y. Therefore (6) is trivial when $x_{n+1}^* = 0$. When $x_{n+1}^* \neq 0$, (6) is easily derived in the case where $\xi \leq t_3$ or $t_{n+2} \leq \xi$. In the case of $t_3 < \xi < t_{n+2}$, it can be shown that zero vector is the unique critical direction. Indeed, we see by Karlin–Ziegler's theorem that all $(n+3) \times (n+3)$-submatrices of the coefficient matrix of (5) are non-singular. Hence all λ_i's are positive. On the other hand, it holds for any critical direction y that

$$0 = S'(x^*;y) = \max\{-\sigma(t)F_x(x^*,t)y;\, t \in T(x^*)\}; \tag{9}$$

see, e.g., Girsanov [4, Ex. 7.5]. Hence, we have

$$\sigma(t_i)F_x(x^*,t_i)y \geq 0, \qquad i = 1,\ldots,n+4. \tag{10}$$

Combining (5) and (10), we have $\lambda_i\sigma(t_i)F_x(x,t_i)y = 0$ for all i. Since $\operatorname{rank}\{F_x(x^*,t_i);\, i = 1,\ldots,n+4\} = n+3$, we get that $y = 0$.

THEOREM 2. *Suppose that Braess's condition is satisfied at x^*. Then*

$$0 \leq \bar{S}''(x^*;y) \leq +\infty \tag{11}$$

for any critical direction y. Moreover, if the error function is expanded into Taylor series as

$$r(t) = \alpha(t-\tau)^k + o(|t-\tau|^k) \qquad \text{for some } \alpha \neq 0,\ k > 0, \tag{12}$$

at its zero points, then $0 \leq S''(x^*;y) \leq +\infty$.

Proof. Let y be any critical direction. It follows from Theorem 1 that

$$-\sigma(\tau)y^TF_{xx}(x^*,\tau)y \geq 0 \tag{13}$$

for some $\tau \in T(x^*)$ such that $F_x(x^*,\tau)y = 0$; see the argument after (10). Next, we put

$$u(t,\sigma) := S(x^*) - \sigma\{f(t) - F(x^*,t)\}, \qquad v(t,\sigma) := \sigma F_x(x^*,t)y, \tag{14}$$

and we denote by $\mathscr{T}_0$ the set of all $(t,\sigma) \in [0,1] \times \{1,-1\}$ for which there exists a sequence $(t_n, \sigma_n) \in [0,1] \times \{1, -1\}$ converging to (t, σ) such that

$$u(t_n, \sigma_n) > 0 \ \forall n, \qquad \lim_{n \to +\infty} -\frac{v(t_n, \sigma_n)}{u(t_n, \sigma_n)} = +\infty. \tag{15}$$

We define a function E: $[0,1] \times \{1, -1\} \to [-\infty, +\infty]$ as follows:

$$E(t,\sigma) := \begin{cases} \sup\left\{ \limsup \dfrac{v(t_n,\sigma_n)^2}{4u(t_n,\sigma_n)};\ \{(t_n,\sigma_n)\} \text{ satisfies } (15) \right\}, \\ \qquad\qquad \text{if } (t,\sigma) \in \mathscr{T}_0, \\ 0, \qquad \text{if } u(t,\sigma) = v(t,\sigma) = 0 \text{ and } (t,\sigma) \notin \mathscr{T}_0, \\ -\infty, \qquad \text{otherwise} \end{cases} \tag{16}$$

Then it follows from Theorem 2.2. in Kawasaki [6] that

$$\bar{S}''(x;y) = \max\left\{ -\tfrac{1}{2}\sigma(t) y^{\mathrm{T}} F_{xx}(x,t) y + E(t,\sigma(t));\ F_x(x^*,t)y = 0 \right\}. \tag{17}$$

Since $u(\tau, \sigma(\tau)) = v(\tau, \sigma(\tau)) = 0$, we have $0 \le E(t, \sigma(t)) \le +\infty$. Hence,

$$-\sigma(\tau) y^{\mathrm{T}} F_{xx}(x^*, \tau) y + 2E(\tau, \sigma(\tau)) \ge 0. \tag{18}$$

Combining (17) and (18), we get (11). Furthermore, it was shown in Theorem 4.3 of [6] that $\bar{S}''(x;y) = S''(x;y)$ if $r(t)$ satisfies the assumption (12).

We close this paper with noting Mulansky's results [7]. Recently, Mulansky improved Braess's condition by utilizing Cromme's results [2, 3] on a tangent cone. Mulansky's condition is in general superior to Braess' condition. However, in our simple context, Mulansky's condition coincides with Braess' condition, because the tangent cone is a subspace.

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